(NOT AN) INTRODUCTION TO TOPOLOGICAL MODULAR FORMS

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ABSTRACT. These are the notes for an introductory talk on topological modular forms given at Dan Berwick-Evans's learning seminar in Fall 2023 at UIUC.

CONTENTS

1. BACKGROUND AND MOTIVATION

Physicists care about *elliptic genera*: ring maps $\Omega_\bullet^{SO} \to \mathbb{C}$ which are multiplicative with respect to bundles of spin manifolds having compact connected structure group.

Theorem 1.1 (Ochanine [8]). *There is a "universal elliptic genus" $\Phi_{univ} : \Omega_\bullet^{SO} \to MF_0(2) \otimes \mathbb{C}$. That is, for any point $p \in \mathbb{H}/\Gamma_0(2)$, $\mathrm{ev}_p \circ \Phi_{univ}$ is an elliptic genus, and this gives a bijective correspondence between elliptic genera and complex elliptic curves with $\Gamma_0(2)$-structure. Moreover, if M is spin, $\Phi(M) \in \mathbb{Z}[[q]]$, i.e. it has integral Fourier coefficients.*

Witten generalized this ([10]). The *generalized Witten genus* of a superstring is the large-volume limit of its partition function. For the type II superstring, this gives Φ_{univ}. For the heterotic superstring, this gives the "ordinary" Witten genus $\Phi : \Omega_\bullet^{SO} \to MF$, which is valued in $\mathbb{Z}[[q]]$ on string manifolds. (If one writes out the formula for these two genera, the only difference is that Ochanine's genus has level 2 Eisenstein series where Witten's has ordinary ones.)

This is all well and good, but why stop here? The genus of a spinning particle lifts to a map of E_∞-rings $MSpin \to KO$ ([2],[4]) which has great geometric significance, replacing a difference of dimensions of spaces with the formal difference of the spaces themselves. Maybe something similar is true here.

Conjecture 1.2 (Witten). *There is an E_∞-ring of "topological modular forms", tmf, and an orientation $MString \to tmf$ lifting Φ.*

Theorem 1.3 (Ando-Hopkins-Rezk [1]). *Yep.*

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In between this conjecture and its resolution, a lot of people worked very hard to construct tmf. To understand how, let's review the chromatic perspective on K-theory.

2. K-Theory, the Chromatic Way

Definition 2.1. The *multiplicative formal group law* over a ring R is given by $F_{mult}(x, y) = x + y + xy$. The associated formal group is called $\widehat{\mathbb{G}}_m$, since it is the completion of $\mathbb{G}_m = \mathrm{Spec} R[t^{\pm}]$.

This is, famously, the formal group of K-theory. But how do we go the other way? Well, the most classical way of constructing a cohomology theory from a formal group law is the Landweber Exact Functor Theorem.

Theorem 2.2 (LEFT)**.** *Suppose a formal group over a stack S is classified by $\varphi : S \to \mathcal{M}_{fg}$, and denote by $Q : \mathrm{Spaces} \to \mathrm{qCoh}(\mathcal{M}_{fg})$ the sheaf-valued homology theory sending X to the sheaf associated to $MU_*(X)$ over $MU_*//MU_*MU \cong \mathcal{M}_{fg}$. If φ is flat, then $\varphi^*(Q)$ is a sheaf of homology theories on S, so the presheaf of ring spectra on $\mathcal{M}_{fg}$ pulls back to a presheaf of ring spectra on S.*

Proof. Clearly $\varphi^*(Q)(X)$ is a quasicoherent sheaf, so we just need to show that this functor is a homology theory. Additivity is clear. Since φ is flat, φ^* preserves exact sequences, and thus the LES axiom holds and this is a homology theory. The last statement now follows from Brown representability. □

Using the explicit version of Landweber's criterion, one can easily check that $\widehat{\mathbb{G}}_m$ is Landweber-exact. Applying this to the completion map $\mathrm{Spec}(\mathbb{Z})//C_2 \cong B\mathbb{G}_m \to \mathcal{M}_{fg}$, we get a presheaf of ring spectra on $B\mathbb{G}_m$ with $\Gamma(B\mathbb{G}_m) \simeq KO$ and $\Gamma(\mathrm{Spec}(\mathbb{Z})) \simeq KU$.

We have just constructed the homotopy-commutative C_2-ring spectrum $K\mathbb{R}$ from just the multiplicative formal group. If we had instead used the additive formal group, we would have gotten ordinary homology. What other formal groups are there? At height ∞, we just have $\widehat{\mathbb{G}}_a$. At height 1, though, in addition to $\widehat{\mathbb{G}}_m$, we have the completions of ordinary elliptic curves, and at height 2 we have the completions of supersingular elliptic curves. So we'd probably do well to see what happens in the elliptic case. Indeed, there are some elliptic curves whose completions are Landweber-exact, and the associated ring spectra are called *elliptic spectra*, with the represented cohomology theories called *elliptic cohomology*. This isn't enough, though. We want to understand the *universal* elliptic cohomology theory; and we want E_∞-rings.

3. A Bit of Spectral Algebraic Geometry

An introduction to the material of this section can be found in [9], and the original source for most of it is [6].

Definition 3.1. A map $A \to B$ of E_∞-rings is called étale if

i) $\pi_0 A \to \pi_0 B$ is étale in the classical sense, and
ii) $A \to B$ is flat, i.e. $\pi_n A \otimes_{\pi_0 A} \pi_0 B \to \pi_n B$ is an isomorphism for all n.

An *étale cover* of A, then, is an étale map $A \to B$ such that $\pi_0 A \to \pi_0 B$ is faithfully flat.

Definition 3.2. The *(small) étale site* of an E_∞-ring R is the category of étale maps $R \to S$ with Grothendieck topology given by étale covers. The topos of sheaves on this site is called $\mathrm{Sp\acute{e}t}(A)$, the *étale spectrum* of A.

Proposition 3.3. $\mathrm{Sp\acute{e}t}(A)$ *carries a canonical sheaf of E_∞-rings in the obvious way.*

Proof. Exercise. ☐

A spectrally-ringed topos is called a *nonconnective spectral Deligne-Mumford stack* if it is locally of the form $\mathrm{Sp\acute{e}t}(A)$. (The term "spectral Deligne-Mumford stack" is traditionally reserved for connective objects.) For convenience, I will abbreviate "nonconnective spectral Deligne-Mumford stack" to "DM-stack" in this talk.

There is a natural notion of morphism of DM-stacks, analogous to the locality condition for morphisms of classical schemes, called a *strictly Henselian morphism.* This yields an adjunction

$$\mathrm{Sp\acute{e}t} : \mathrm{CAlg} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{DM} : \Gamma(-)$$

as we would expect.

As in the classical case, we have a notion of *formal DM-stack*, which includes the étale spectra of adic E_∞-rings (E_∞-rings with an adic topology on π_0). With this comes a notion of formal group, a group object in "formal hyperplanes" (the formal DM analogue of affine space). As usual, we take all formal groups to be one-dimensional and commutative. It will be convenient to represent adic E_∞-rings by their duals, which are E_∞-coalgebras; see [5] for more details. Suffice for now to say that this corresponds to working with homology rather than cohomology, and involves no loss of information for reasons of dualizability.

4. Oriented Elliptic Curves

Recall that for any complex-oriented ring spectrum E, we get a classical formal group $E^*(\mathbb{CP}^\infty)$. Here is the E_∞ version of that construction.

Definition 4.1. Let R be a complex-periodic[1] E_∞-ring. The *Quillen formal group* of R is the spectral formal group $\widehat{G}^Q_R$ represented by the smooth E_∞-coalgebra $R \otimes_{\mathbb{S}} \Sigma^\infty_+ \mathbb{CP}^\infty$ with group structure coming from the usual operation on $\mathbb{CP}^\infty$.

Definition 4.2. Let G be a formal group over R. An *orientation* of G is an isomorphism $\widehat{G}^Q_R \simeq G$.

For example, consider $R = KU$. Then $\widehat{G}^Q_R \simeq \widehat{\mathbb{G}}_m$, where the right hand side is the formal multiplicative group of KU. More generally, we can consider the *moduli stack of oriented multiplicative groups*, defined by the property $\mathrm{DM}(\mathrm{Sp\acute{e}t}R, \mathcal{M}^{or}_{\mathbb{G}_m}) \simeq \mathrm{Iso}(\widehat{G}^Q_R, \widehat{\mathbb{G}}_m(R))$. This will give us $K\mathbb{R}$ as before, but now as a C_2-E_∞-ring. To get the height 2 analogue of this construction, we can replace oriented $\mathbb{G}_m$s with oriented elliptic curves.

[1] Short for "complex-oriented and even-periodic". While a complex-oriented ring spectrum is one equipped with a homotopy-commutative ring map from MU, a complex-periodic ring spectrum is equivalently one equipped with a homotopy-commutative ring map from periodic complex cobordism MUP.

Definition 4.3. A *strict abelian variety* is a product-preserving functor $\mathrm{Lat}^{op} \to \mathrm{DM}$, where Lat is the category of finite-rank free abelian groups. (That is, it is a model for the Lawvere theory of abelian groups.) An *oriented elliptic curve* is a one-dimensional strict abelian variety together with an orientation of its completion.

Recall that classically, we can define integral modular forms as the graded ring of sections of powers of a line bundle ω over the moduli stack of elliptic curves, $MF_N = \Gamma(\mathcal{M}_{ell}; \omega^{\otimes N})$. (To be precise, ω is the pullback of the invertible sheaf of fiberwise Kähler differentials of the universal elliptic curve.) We use a similar method to construct tmf.

Theorem 4.4 (Goerss-Hopkins-Miller-Lurie [5])**.** *Let $\mathcal{U}$ be the étale topos of the classical moduli stack of elliptic curves, and $\mathcal{O} : \mathcal{U}^{op} \to \mathrm{CAlg}^{\heartsuit}$ its structure sheaf of classical rings. Then $\mathcal{O}$ lifts uniquely to a sheaf of E_∞-rings $\mathcal{O}^{top}$ on $\mathcal{U}$ with $\pi_0(\mathcal{O}^{top}) \cong \mathcal{O}$ such that for any elliptic curve C with étale classifying map $u : \mathrm{Spét} A \to \mathcal{M}_\ell$,*

i) $\mathcal{O}^{top}(u)$ is complex-periodic, and

ii) The classical formal group of $\mathcal{O}^{top}(u)$ is naturally ismorphic to $\widehat{C}$.

The spectrally-ringed topos $\mathcal{M}_{ell}^{or} := (\mathcal{U}, \mathcal{O}^{top})$ is a DM-stack, and it is the moduli stack of oriented elliptic curves.

This theorem can be proven either by direct application of E_∞ obstruction theory (the method of Goerss-Hopkins-Miller) or by working with deformations of p-divisible groups (the method of Lurie).

Both $\mathcal{M}_{\mathbb{G}_m}^{or}$ and $\mathcal{M}_{ell}^{or}$ admit maps to $\mathcal{M}_{fg}^{or}$, the moduli stack of oriented formal groups. (This last stack is probably not Deligne-Mumford, so we probe it using DM-stacks.)

Definition 4.5. $TMF = \Gamma(\mathcal{M}_{ell}^{or}, \mathcal{O}^{top})$ is called *periodic tmf*. If we replace this stack with its Deligne-Mumford compactification $\widehat{\mathcal{M}}_{ell}^{or}$, the resulting E_∞-ring is called Tmf (compactified tmf); and the connective cover of compactified tmf is called tmf (connective tmf).

Nice Facts:

- TMF is 576-periodic.
- $\pi_*(tmf)$ is close to MF, but not exactly equal.
- There is an E_∞ orientation $MString \to tmf$ lifting the Witten genus.
- $\pi_1^{ét}(\mathcal{M}_{ell}^{or}) \cong 1$. However, $\mathcal{M}_{ell}^{or}[N^{-1}]$ has étale covers whose global sections are things like $TMF(N)$, $TMF_0(N)$, and $TMF_1(N)$. "Tmf with level structure" is interesting and complicated, and can be used to understand genuine C_n-equivariant tmf ([7]). If we go all the way and look at $\mathcal{M}_{ell}^{or} \otimes \mathbb{C}$, we get an étale cover for every congruence subgroup $\Gamma \subset GL_2(\mathbb{Z})$. Describing modular-equivariant tmf is an ongoing project; the state of the art is [3], which generalizes this to Tmf (and tmf) by looking at log-étale covers of the compactified moduli stack.

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